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by

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This work was started when the second author was at the Indian Statistical Institute, New Delhi.

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INTRODUCTION

Almost twenty years ago Singh [7] and Tracy and Singh [9] introduced a generalization of the Kronecker product $A \otimes B$. They used varying notation for this new product, viz. $A \pi B$ and $A \square B$. Recently, Hyland and Collins [1] studied the same product under rather restrictive order conditions. They denoted it $A \bar{\otimes} B$. Curiously enough, they were not aware of their predecessors' work.

Tracy and Jinadasa [8] considered the product again very recently. They elected to use the notation $A \pi B$.

Closely related to the new product is a generalization of the vec operator, which Hyland and Collins very aptly named 'vecb' operator. This and another very similar operator can also be found in the work of the other authors. Hyland and Collins integrated the new product and the vecb operator in a nice way.

In the present paper we define a less restricted generalized Kronecker product. Whereas the product studied before is based on the partitioning of both matrices A and B, ours presupposes a partition of B solely. Researchers can partition A according to their needs and wishes. This approach is mathematically less cumbersome and very suitable for certain unbalanced partitionings. We propose to use the notation $A \square B$ for this generalized Kronecker product. The other product will be denoted $A \pi B$. Both products will be called 'block Kronecker products'.

The term 'vecb' operator will be used to cover the two variants of the generalized vec operator. These will be distinguished notationally as vecb_c and vecb_r and will be shown to be compatible with the two above mentioned block Kronecker products.

In the first section of this paper we discuss the case of balanced partitioning. This involves the product $A \pi B$, the vecb_c operator and the so-called tilde transform.

In the second section we elaborate the general case of unbalanced partitioning, where we introduce the block Kronecker product $A \square B$ and the compatible vecb_r operator.

In the third section we consider linear equations in partitioned matrices.

Finally, we apply some of the techniques discussed in the fourth and last section.

1 BALANCED PARTITIONING

1.1 The vecb_c operator, the tilde transform and the balanced block Kronecker product $A\pi B$

Throughout section 1 the matrix A is a blockwise partitioned matrix, $A = (A_{ij})$, where A_{ij} is of order $m\times n$ ($i = 1,..,p$; $j = 1,..,q$). So A is of order $mp\times nq$, and we can write

$$A = \sum_{ij} \left(E_{ij}\otimes A_{ij}\right), \tag{1}$$

where the matrix E_{ij} is of order $p\times q$, having a unit element in the (i,j)th position and zeros elsewhere. Obviously, the order of E_{ij} corresponds to the partition of A. We define

$$\text{vecb}_c A \equiv \sum_{ij} \left(\text{vec}E_{ij}\otimes\text{vec}A_{ij}\right). \tag{2}$$

Thus, if $p = q = 2$, so

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

then

$$\text{vecb}_c A = \begin{pmatrix} \text{vec}A_{11} \\ \text{vec}A_{21} \\ \text{vec}A_{12} \\ \text{vec}A_{22} \end{pmatrix}.$$

We have vectorized the partitioned matrix A column-blockwise. This is only one way of vectorizing the partitioned matrix A, in agreement with the literature (Tracy and Singh [9], Singh [7], Tracy and Jinadasa [8], Hyland and Collins [1]). However, we can also vectorize row-blockwise, *i.e.*, vectorize each row of blocks, and stack the resulting vectors. This leads to another type of vecb operator:

$$\text{vecb}_r A = \sum_{ij} \left(\text{vec}E_{ji}\otimes\text{vec}A_{ij}\right) \tag{3}$$

and for the example we have

$$\text{vecb}_r A = \begin{pmatrix} \text{vec}A_{11} \\ \text{vec}A_{12} \\ \text{vec}A_{21} \\ \text{vec}A_{22} \end{pmatrix}.$$

For the rest of this section we only consider the vecb_c operator. We consider the vecb_r operator again in section 2, where we discuss the case of unbalanced partitioning.

If we denote the commutation matrix (see Magnus and Neudecker [4]) by K, then the following property is easily established:

$$\text{vecb}_c A = \left(I_q \otimes K_{pn} \otimes I_m\right) \text{vec}A, \tag{4}$$

since

$$\begin{aligned} \text{vecb}_c A &= \textstyle\sum_{ij} \left(\text{vec}E_{ij} \otimes \text{vec}A_{ij}\right) = \textstyle\sum_{ij} \left(I_q \otimes K_{pn} \otimes I_m\right) \text{vec}\left(E_{ij} \otimes A_{ij}\right) \\ &= \left(I_q \otimes K_{pn} \otimes I_m\right) \textstyle\sum_{ij} \text{vec}\left(E_{ij} \otimes A_{ij}\right) = \left(I_q \otimes K_{pn} \otimes I_m\right) \text{vec}A, \end{aligned}$$

by virtue of the result

$$\text{vec}\left(Y \otimes Z\right) = \left(I_q \otimes K_{np} \otimes I_m\right) \left(\text{vec}Y \otimes \text{vec}Z\right) \tag{5}$$

for matrices Y $(p \times q)$ and Z $(m \times n)$ (see Neudecker and Wansbeek [6]).

Closely connected with this is the so-called tilde transform, which links the vec operator and the vecb_c operator. The tilde transform of A is defined by

$$\tilde{A} = \textstyle\sum_{ij} \left(\text{vec}A_{ij}\right) \left(\text{vec}E_{ij}\right)'. \tag{6}$$

For example, if again $p = q = 2$, we get

$$\tilde{A} = \left(\text{vec}A_{11},\ \text{vec}A_{21},\ \text{vec}A_{12},\ \text{vec}A_{22}\right).$$

From (4), (5) and (6) we have

$$\text{vec}\tilde{A} = \text{vecb}_c A = \left(I_q \otimes K_{pn} \otimes I_m\right) \text{vec}A. \tag{7}$$

We next consider the equation

$$\text{vecb}_c BCA' = X \text{vecb}_c C, \tag{8}$$

for A as defined above,

$$B = (B_{kl}) = \sum_{kl} \left(E_{kl} \otimes B_{kl}\right),$$

with B_{kl} of order $u \times v$ $(k = 1,..,s;\ l = 1,..,t)$ and

$$C = (C_{lj}) = \sum_{lj} \left(E_{lj} \otimes C_{lj}\right),$$

with C_{lj} of order $v \times n$ $(l = 1,..,t;\ j = 1,..,q)$. Using (4) we have

$$\begin{aligned} \mathrm{vecb}_c BCA' &= \left(I_p \otimes K_{sm} \otimes I_u\right) \mathrm{vec} BCA' = \left(I_p \otimes K_{sm} \otimes I_u\right) \left(A \otimes B\right) \mathrm{vec} C \\ &= \left(I_p \otimes K_{sm} \otimes I_u\right) \left(A \otimes B\right) \left(I_q \otimes K_{nt} \otimes I_v\right) \mathrm{vecb}_c C, \end{aligned} \tag{9}$$

from which follows the solution

$$\mathrm{vecb}_c BCA' = \left(A \pi B\right) \mathrm{vecb}_c C, \tag{10}$$

with

$$A \pi B \equiv \left(I_p \otimes K_{sm} \otimes I_u\right) \left(A \otimes B\right) \left(I_q \otimes K_{nt} \otimes I_v\right). \tag{11}$$

This defines the balanced block Kronecker product $A\pi B$ (*cf.* Tracy and Singh [9] and Hyland and Collins [1], formula (2.4)). If $p = q = s = 2$ and $t = 3$, the block Kronecker product of A and B takes the form

$$A \pi B = \begin{pmatrix} A_{11} \otimes B_{11} & A_{11} \otimes B_{12} & A_{11} \otimes B_{13} & A_{12} \otimes B_{11} & A_{12} \otimes B_{12} & A_{12} \otimes B_{13} \\ A_{11} \otimes B_{21} & A_{11} \otimes B_{22} & A_{11} \otimes B_{23} & A_{12} \otimes B_{21} & A_{12} \otimes B_{22} & A_{12} \otimes B_{23} \\ A_{21} \otimes B_{11} & A_{21} \otimes B_{12} & A_{21} \otimes B_{13} & A_{22} \otimes B_{11} & A_{22} \otimes B_{12} & A_{22} \otimes B_{13} \\ A_{21} \otimes B_{21} & A_{21} \otimes B_{22} & A_{21} \otimes B_{23} & A_{22} \otimes B_{21} & A_{22} \otimes B_{22} & A_{22} \otimes B_{23} \end{pmatrix}.$$

If A is partitioned elementwise (*i.e.*, $m = n = 1$), the vecb_c operator and the block Kronecker product reduce to the usual vec operator and Kronecker product.

This derivation of the balanced block Kronecker product $A\pi B$ is inspired by Hyland and Collins. We prefer it to the explicit definitions of Tracy and Singh and Singh and Jinadasa, since this derivation shows how the product arises in a natural manner from vectorizing a partitioned matrix. We can write $A\pi B$ in a form that elegantly generalizes (1):

$$A\pi B = \sum_{ijkl}\left\{\left(E_{ij}\otimes E_{kl}\right)\otimes\left(A_{ij}\otimes B_{kl}\right)\right\}, \tag{11'}$$

because

$$\begin{aligned} A\pi B &= \left(I_p\otimes K_{sm}\otimes I_u\right)\left(A\otimes B\right)\left(I_q\otimes K_{nt}\otimes I_v\right) \\ &= \sum_{ijkl}\left(I_p\otimes K_{sm}\otimes I_u\right)\left(E_{ij}\otimes A_{ij}\otimes E_{kl}\otimes B_{kl}\right)\left(I_q\otimes K_{nt}\otimes I_v\right) \\ &= \sum_{ijkl}\left\{E_{ij}\otimes K_{sm}\left(A_{ij}\otimes E_{kl}\right)K_{nt}\otimes B_{kl}\right\} \\ &= \sum_{ijkl}\left\{\left(E_{ij}\otimes E_{kl}\right)\otimes\left(A_{ij}\otimes B_{kl}\right)\right\}. \end{aligned}$$

Here the orders of the matrices E_{ij} and E_{kl} correspond to the respective partitions of A and B. Hence E_{ij} is of order $p\times q$ and E_{kl} is of order $s\times t$.

1.2 Properties of the vecb$_c$ operator and the balanced block Kronecker product $A\pi B$

1. Let $a = (a_i)$ and $b = (b_k)$, where a_i is of order $m\times 1$ and b_k is of order $u\times 1$ ($i = 1,..,p$; $k = 1,..,s$), then

$$a\pi b = \mathrm{vecb}_c ba' = \left(I_p\otimes K_{sm}\otimes I_u\right)\left(a\otimes b\right). \tag{12}$$

Proof: Replace A by a, B by b and n, q, v, t by 1 in (11). Replace C by 1 in (10).

2. Let A and B be defined as above. Then

$$\mathrm{vecb}_c\left(A\pi B\right) = \left(I_q\otimes K_{tp}\otimes I_{sn}\otimes K_{vm}\otimes I_u\right)\left(\mathrm{vecb}_c A\pi\mathrm{vecb}_c B\right) \tag{13}$$

Proof:

$$\begin{aligned} \mathrm{vecb}_c A\pi\mathrm{vecb}_c B &= \left(I_{pq}\otimes K_{st,mn}\otimes I_{uv}\right)\left(\mathrm{vecb}_c A\otimes\mathrm{vecb}_c B\right) \\ &= \left(I_{pq}\otimes K_{st,mn}\otimes I_{uv}\right)\sum_{ijkl}\left(\mathrm{vec}E_{ij}\otimes\mathrm{vec}A_{ij}\otimes\mathrm{vec}E_{kl}\otimes\mathrm{vec}B_{kl}\right) \\ &= \sum_{ijkl}\left(\mathrm{vec}E_{ij}\otimes\mathrm{vec}E_{kl}\otimes\mathrm{vec}A_{ij}\otimes\mathrm{vec}B_{kl}\right), \end{aligned}$$

by virtue of (2) and (11); furthermore,

$$I_q\otimes K_{tp}\otimes I_{sn}\otimes K_{vm}\otimes I_u = \left(I_q\otimes K_{tp}\otimes I_s\right)\otimes\left(I_n\otimes K_{vm}\otimes I_u\right),$$

hence we find

$$\left(I_q \otimes K_{tp} \otimes I_{sn} \otimes K_{vm} \otimes I_u\right)\left(\mathrm{vecb}_c A \pi \mathrm{vecb}_c B\right)$$
$$= \sum_{ijkl}\left\{\mathrm{vec}\left(E_{ij} \otimes E_{kl}\right) \otimes \mathrm{vec}\left(A_{ij} \otimes B_{kl}\right)\right\}$$
$$= \mathrm{vecb}_c \sum_{ijkl}\left\{\left(E_{ij} \otimes E_{kl}\right) \otimes \left(A_{ij} \otimes B_{kl}\right)\right\}$$
$$= \mathrm{vecb}_c\left(A \pi B\right),$$

by using (2) and (11').

3. When A is elementwise partitioned ($m = n = 1$), then

$$\mathrm{vecb}_c\left(A \otimes B\right) = \mathrm{vec}A \otimes \mathrm{vec}B. \tag{14}$$

If further $C = A \otimes B$, then

$$\tilde{C} = \left(\mathrm{vec}B\right)\left(\mathrm{vec}A\right)'. \tag{15}$$

Proof:

$$A = \sum_{ij} a_{ij} E_{ij},$$

hence

$$\mathrm{vecb}_c(A \otimes B) = \sum_{ij} a_{ij} \mathrm{vecb}_c\left(E_{ij} \otimes B\right) = \sum_{ij} a_{ij}\left(\mathrm{vec}E_{ij} \otimes \mathrm{vec}B\right) =$$
$$\mathrm{vec}A \otimes \mathrm{vec}B$$

by (2).

Further,

$$\tilde{C} = \sum_{ij}\left(\mathrm{vec}C_{ij}\right)\left(\mathrm{vec}E_{ij}\right)' = \sum_{ij}\left(\mathrm{vec}a_{ij}B\right)\left(\mathrm{vec}E_{ij}\right)'$$
$$= \left(\mathrm{vec}B\right)\left(\sum_{ij} \mathrm{vec}a_{ij}E_{ij}\right)' = \left(\mathrm{vec}B\right)\left(\mathrm{vec}A\right)'.$$

4. When A and B are columnwise partitioned ($n = v = 1$), then

$$\mathrm{vecb}_c\left\{\left(\mathrm{vec}B\right)\left(\mathrm{vec}A\right)'\right\} = \mathrm{vec}\left(A \otimes B\right). \tag{16}$$

If further $C = (\mathrm{vec}B)(\mathrm{vec}A)'$, then

$$\tilde{C} = A \otimes B. \tag{17}$$

Proof:

$$\mathrm{vecb}_c\{(\mathrm{vec}B)(\mathrm{vec}A)'\} = (I_q \otimes K_{t,mp} \otimes I_{us})(\mathrm{vec}A \otimes \mathrm{vec}B) = \mathrm{vec}A \otimes B,$$

by virtue of (12) and (5). Finally,

$$C = \sum_{ij} \{E_{ij} \otimes (B_{.i}A_{.j}')\}$$

$$\begin{aligned} \tilde{C} &= \sum_{ij} \{[\mathrm{vec}(B_{.i}A_{.j}')][\mathrm{vec}E_{ij}]'\} \\ &= \sum_{ij} (A_{.j} \otimes B_{.i})(e_j' \otimes e_i') \\ &= \left[\sum_j A_{.j}e_j'\right] \otimes \left[\sum_i B_{.i}e_i'\right] \\ &= A \otimes B, \end{aligned}$$

by (1) and (6).

5. Let A, B, C, D be partitioned matrices of order $mp \times nq$, $nq \times rk$, $rk \times sl$ and $mp \times sl$ whose blocks are of order $m \times n$, $n \times k$, $k \times l$ and $l \times m$, respectively. Then

$$\mathrm{tr}ABCD' = (\mathrm{vec}A')'(D \otimes B)\mathrm{vec}C = (\mathrm{vecb}_c A')'(D \pi B)\mathrm{vecb}_c C \tag{18}$$

Proof:

$$\begin{aligned} \mathrm{tr}ABCD' &= (\mathrm{vec}A')'(D \otimes B)\mathrm{vec}C \\ &= (\mathrm{vec}A')'(I_p \otimes K_{mq} \otimes I_n)(I_p \otimes K_{qm} \otimes I_n)(D \otimes B) \cdot \\ &\qquad (I_s \otimes K_{lr} \otimes I_k)(I_s \otimes K_{rl} \otimes I_k)\mathrm{vec}C \\ &= (\mathrm{vecb}_c A')'(D \pi B)\mathrm{vecb}_c C, \end{aligned}$$

by virtue of (11).

6. When A and B are columnwise partitioned ($n = v = 1$),

$$A\pi B \;=\; A\otimes B. \tag{19}$$

Proof:

$$A\pi B \;=\; \left(I_1\otimes K_{1,mp}\otimes I_{us}\right)\left(A\otimes B\right)\left(I_q\otimes K_{1t}\otimes I_1\right) \;=\; A\otimes B,$$

by (11).

In section 2.2 we prove sixteen properties of the block Kronecker product in the unbalanced case. *Mutatis mutandis*, the same properties hold for the case of balanced partitioning as discussed here.

2 UNBALANCED PARTITIONING

2.1 The vecb_r operator and the block Kronecker product

Let us consider the blockwise partitioned matrix

$$A \;=\; \begin{bmatrix} A_{11} & . & . & A_{1q} \\ \vdots & & & \vdots \\ A_{p1} & . & . & A_{pq} \end{bmatrix} \;\equiv\; \begin{bmatrix} A_{1.} \\ \vdots \\ A_{p.} \end{bmatrix},$$

with typical block A_{ij} of order $m_i \times n_j$. Furthermore, $m \equiv \sum_{i=1}^{p} m_i$ and $n \equiv \sum_{j=1}^{q} n_j$. So contrary to what we had above, A is now an $m \times n$-matrix, not all submatrices being of the same size. This is an example of a matrix partitioned in an inbalanced way. Various authors have defined the vecb_c operator on this matrix, which they call vecb.

Let us consider the equation

$$\mathrm{vecb}_c BCA' \;=\; X \mathrm{vecb}_c C,$$

with B a $u \times v$ matrix consisting of blocks B_{kl} of order $u_k \times v_l$ ($u = \sum_{k=1}^{s} u_k$, $v = \sum_{l=1}^{t} v_l$) and C an $v \times n$ matrix consisting of blocks $v_l \times n_j$. The solution is then the block Kronecker product $A\pi B$. We have

$$\left(BCA'\right)_{ki} \;=\; \textstyle\sum_{jl} \left(B_{kl}C_{lj}A'_{ji}\right)$$

$$\mathrm{vec}\left(BCA'\right)_{ki} \;=\; \textstyle\sum_{jl} \mathrm{vec}\left(B_{kl}C_{lj}A'_{ji}\right) \;=\; \textstyle\sum_{jl} \left(A_{ij}\otimes B_{kl}\right)\mathrm{vec}C_{lj}$$

$$= \left(A_{i1}\otimes B_{k1},..,A_{i1}\otimes B_{kt},..,A_{iq}\otimes B_{k1},..,A_{iq}\otimes B_{kt}\right) \mathrm{vecb}_c C.$$

Hence

$$\mathrm{vecb}_c BCA' = \begin{bmatrix} A_{11}\otimes B_{11}..A_{11}\otimes B_{1t}....A_{1q}\otimes B_{11}..A_{1q}\otimes B_{1t} \\ \vdots \\ A_{11}\otimes B_{s1}..A_{11}\otimes B_{st}....A_{1q}\otimes B_{s1}..A_{1q}\otimes B_{st} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ A_{p1}\otimes B_{11}..A_{p1}\otimes B_{1t}....A_{pq}\otimes B_{11}..A_{pq}\otimes B_{1t} \\ \vdots \\ A_{p1}\otimes B_{s1}..A_{p1}\otimes B_{st}....A_{pq}\otimes B_{s1}..A_{pq}\otimes B_{st} \end{bmatrix} \mathrm{vecb}_c C$$

$$= \left(A\pi B\right) \mathrm{vecb}_c C.$$

A restrictive feature of this approach is that all three matrices have to be partitioned. However, by vectorizing in a different manner, this problem can be avoided.

Let us therefore consider the equation

$$\mathrm{vecb}_r BCA' = X \mathrm{vecb}_r C, \tag{20}$$

with B as defined above but A is a not necessarily partitioned $m{\times}n$ matrix. Using the notation introduced above, we have as the k^{th} block row of BCA'

$$\left(BCA'\right)_{k.} = \sum_l B_{kl} C_{l.} A' \qquad k = 1,..,s$$

hence,

$$\mathrm{vec}\left(BCA'\right)_{k.} = \sum_l \left(A\otimes B_{kl}\right) \mathrm{vec} C_{l.} = \left(A\otimes B_{k1},.., A\otimes B_{kt}\right) \mathrm{vecb}_r C$$

and consequently

$$\mathrm{vecb}_r BCA' = \begin{bmatrix} A\otimes B_{11} & . \; . & A\otimes B_{1t} \\ \vdots & & \vdots \\ A\otimes B_{s1} & . \; . & A\otimes B_{st} \end{bmatrix} \mathrm{vecb}_r C.$$

The solution turns out to be another block Kronecker product, which we will denote by "$\Box$":

$$X = A\Box B \equiv \begin{bmatrix} A\otimes B_{11} & . \; . & A\otimes B_{1t} \\ \vdots & & \vdots \\ A\otimes B_{s1} & . \; . & A\otimes B_{st} \end{bmatrix}. \tag{21}$$

Note that we did not partition A, since there is no mathematical need to do so. One can partition A if one wants and as one pleases, given the nature of the practical problem one is concerned with. The fact that it is unnecessary to partition A follows from our different way of vectorization. While taking the vecb_c of a matrix A, one makes a "jump" from A_{11} to A_{21}. The vecb_r operator does not make this jump, and in that sense, this operator preserves more of the structure of the matrix A.

2.2 Properties of the block Kronecker product $A\Box B$

We shall prove sixteen properties of the block Kronecker product $A\Box B$, more or less following the set–up of Hyland and Collins. They derive similar properties for the block Kronecker product $A\pi B$. We shall, however, impose less restrictive order conditions than they did. Moreover, the properties of the product $A\Box B$ are much easier to derive than for $A\pi B$, if the partition is unbalanced. The properties are:

1. $\text{vecb}_r BCA' = (A\Box B)\text{vecb}_r C.$

   Proof: Already supplied in Section 2.1.

2. $\text{vecb}_r(AD+DB) = (I\Box A+B'\Box I)\text{vecb}_r D.$

   Proof:

$$\text{vecb}_r(AD+DB) = \text{vecb}_r ADI + \text{vecb}_r IDB = (I\Box A)\text{vecb}_r D$$

$$+ (B'\Box I)\text{vecb}_r D = (I\Box A+B'\Box I)\text{vecb}_r D.$$

3. $A\Box B = K_1(B\otimes A)K_2$ for some permutation matrices K_1 and K_2.

   Proof: Let A be an $m\times n$ matrix and let B be a $u\times v$ matrix, with blocks of order $u_l\times v_k$ ($l = 1,..,r$, $k = 1,..,s$). Then

$$B\otimes A = \begin{bmatrix} B_{11}\otimes A & .. & B_{1s}\otimes A \\ . & & . \\ B_{r1}\otimes A & .. & B_{rs}\otimes A \end{bmatrix} = \begin{bmatrix} K_{u_1m} & & \\ & \ddots & \\ & & K_{u_rm} \end{bmatrix} (A\Box B) \begin{bmatrix} K_{nv_1} & & \\ & \ddots & \\ & & K_{nv_s} \end{bmatrix}.$$

Hence

$$A\Box B = \begin{bmatrix} K_{mu_1} & & \\ & \ddots & \\ & & K_{mu_r} \end{bmatrix} (B\otimes A) \begin{bmatrix} K_{v_1n} & & \\ & \ddots & \\ & & K_{v_sn} \end{bmatrix}.$$

4. $(A+B)\square C = A\square C + B\square C$.

Proof: Evident.

5. $A\square(B+C) = A\square B + A\square C$.

Proof: Evident.

6. $(A\square B)' = A'\square B'$.

Proof: Follows from 3.

7. $(A\square B)(C\square D) = AC\square BD$.

Proof:

$$(A\square B)(C\square D)\text{vecb}_r X = (A\square B)\text{vecb}_r DXC' = \text{vecb}_r\ BDXC'A'$$
$$= \text{vecb}_r BDX(AC)' = (AC\otimes BD)\text{vecb}_r X.$$

As this holds for any X, the result follows.

8. $(A\square B)^- = A^-\square B^-$, for any generalized inverse.

Proof: Follows from 7.

9. $B\square A = K_1(A\square B)K_2$ for some permutation matrices K_1 and K_2.

Proof: Follows from 3.

10. $|A\square B| = |A|^n.|B|^m$ for square matrices A $m\times m$ and B $n\times n$.

Proof: Use 3 to get

$$|A\square B| = |B\otimes A| = |A|^n.|B|^m.$$

Note: The partition is clearly immaterial.

11. $\text{tr}(A\square B) = \text{tr}A.\text{tr}B$ for square matrices A and B and B with square diagonal blocks.

Proof: Let $B_{11},...,B_{nn}$ be the square diagonal blocks of B. Then

$$\text{tr}(A\square B) = \sum_i \text{tr}(A\otimes B_{ii}) = \text{tr}A.(\sum_i \text{tr}B_{ii}) = \text{tr}A.\text{tr}B.$$

12. $(I\square A)(B\square I) = (B\square I)(I\square A) = B\square A$.

Proof: Follows from 7 applied twice.

13. $\exp(A\Box I+I\Box B) = \exp A\Box\exp B.$

Proof:

$$\exp(A\Box I+I\Box B) = \exp(A\Box I)\exp(I\Box B) = \exp\{K(I\otimes A)K'\}\exp\{K(B\otimes I)K'\}$$

$$= \sum_{i=0}^{\infty}\{K(I\otimes A)K'\}^i.\sum_{j=0}^{\infty}\{K(B\otimes I)K'\}^j$$

$$= K\sum_{i=0}^{\infty}(I\otimes A)^iK'.K\sum_{j=0}^{\infty}(B\otimes I)^jK'$$

$$= K\sum_{i=0}^{\infty}(I\otimes A^i).\sum_{j=0}^{\infty}(B^j\otimes I)K' = K(I\otimes\exp A)(\exp B\otimes I)K'$$

$$= K(\exp B\otimes\exp A)K' = \exp A\Box\exp B.$$

14. The matrices $A\Box B$ and $B\otimes A$ have identical sets of eigenvalues, for square A and B.

Proof: Suppose $A(B)$ has eigenvalues $\lambda_i(\mu_j)$ and associated eigenvectors x_i (y_j). Then

$$(B\otimes A)(y_j\otimes x_i) = \lambda_i\mu_j(y_j\otimes x_i).$$

Hence, by virtue of 3,

$$K_1(A\Box B)K_2(y_j\otimes x_i) = K_1(A\Box B)(x_i\Box y_j) = K_1(Ax_i\Box By_j)$$

$$= \lambda_i\mu_jK_1(x_i\Box y_j),$$

or

$$(A\Box B)(x_i\Box y_j) = \lambda_i\mu_j(x_i\Box y_j).$$

15. The matrices $A\Box I + I\Box B$ and $I\Box A + B\Box I$ have identical sets of eigenvalues.

Proof: Let A and B be defined as above. Then

$$(I\otimes A+B\otimes I)(y_j\otimes x_i) = (\lambda_i+\mu_j).(y_j\otimes x_i).$$

Hence

$$K_1'(A\Box I+I\Box B)K_2'(y_j\otimes x_i) = K_1'(A\Box I+I\Box B)(x_i\Box y_j)$$

$$= K_1'(Ax_i\Box y_j + x_i\Box By_j) = (\lambda_i+\mu_j)K_1'(x_i\Box y_j),$$

or

$$(A\Box I + I\Box B)(x_i\Box y_j) \quad = \quad (\lambda_i+\mu_j)(x_i\Box y_j).$$

16. When A has eigenvectors x_i and B has eigenvectors y_j, then $A\Box B$ and $A\Box I$ + $I\Box B$ will have eigenvectors $x_i\Box y_j$.
    Proof: Has been supplied under 14 and 15.

As stated above, Hyland and Collins presented identical properties for the vecb_c operator. Unfortunately, their block Kronecker product was unnecessarily constrained as to its partitioning. Tracy and Singh proved the properties 4, 5, 11 and 12 without unnecessarily constraining their matrices.

3 THE EQUATION vec*X* = vecb$_r$*C*.

We shall now consider the equation

$$\mathrm{vec}X \quad = \quad \mathrm{vecb}_r C. \tag{22}$$

When C is appropriately partitioned we can solve for X. The solution is not always unique, even if we preserve the partition of C. Take for example

$$C \quad = \quad \begin{array}{c c} n & \\ \begin{pmatrix} C_1 \\ \cdot \\ \cdot \\ C_p \end{pmatrix} & \begin{array}{c} m_1 \\ \\ \\ m_p \end{array} \end{array} .$$

Then we have the solutions

$$X_1 \quad = \quad (\mathrm{vec}C_1,...,\mathrm{vec}C_p)$$

and

$$X_2 \quad = \quad (C_1,...,C_p),$$

because

$$\mathrm{vecb}_r C \quad = \quad \begin{pmatrix} \mathrm{vec}C_1 \\ \cdot \\ \cdot \\ \mathrm{vec}C_n \end{pmatrix} \quad = \quad \mathrm{vec}(\mathrm{vec}C_1,...,\mathrm{vec}C_p) \quad = \quad \mathrm{vec}(C_1,...,C_p).$$

Another example is

$$C = \begin{array}{cc} n & n \\ \begin{bmatrix} C_{11} & \cdot & C_{1q} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ C_{p1} & \cdot & C_{pq} \end{bmatrix} & \begin{array}{c} m \\ \\ \\ m \end{array} \end{array},$$

where all blocks have the same order ($m\times n$). Then we have one solution

$$X = (\text{vec}C_{11},..,\text{vec}C_{1q},..,\text{vec}C_{p1},..,\text{vec}C_{pq}),$$

because

$$\text{vecb}_r C = \begin{bmatrix} \text{vec}C_1 \\ \cdot \\ \cdot \\ \text{vec}C_p \end{bmatrix} = \begin{bmatrix} \text{vec}(C_{11},\dots,C_{1q}) \\ \cdot \\ \cdot \\ \text{vec}(C_{p1},\dots,C_{pq}) \end{bmatrix}$$

$$= \begin{bmatrix} \text{vec}C_{11} \\ \text{vec}C_{1q} \\ \text{vec}C_{p1} \\ \text{vec}C_{pq} \end{bmatrix} = \text{vec}(\text{vec}C_{11},..,\text{vec}C_{1q},..,\text{vec}C_{p1},..,\text{vec}C_{pq}).$$

A third example is

$$C = \begin{bmatrix} c_{11} & \cdot & c_{1q} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ c_{p1} & \cdot & c_{pq} \end{bmatrix},$$

the matrix being partitioned elementwise.
Now we have the solution

$$X = C',$$

because

$$\text{vecb}_r C = (c_{11},..,c_{1q},..,c_{p1},..,c_{pq})' = \text{vec}C'.$$

We have the following two results. First, if $C = A\otimes B$, where A is elementwise partitioned, then

$$X_1 = (\text{vec}B)(\text{vec}A)', \tag{23a}$$

$$X_2 = (\text{vec}A')'\otimes B \tag{23b}$$

are solutions of the equation $\mathrm{vec}X = \mathrm{vecb}_r C$. Since

$$C = A \otimes B = \begin{pmatrix} a_{11}B & . & .a_{1n}B \\ . & & . \\ . & & . \\ a_{m1}B & . & .a_{mn}B \end{pmatrix},$$

say, we have

$$\mathrm{vecb}_r C = \begin{pmatrix} a_{11}\mathrm{vec}B \\ a_{1n}\dot{\mathrm{v}}\mathrm{ec}B \\ a_{m1}\dot{\mathrm{v}}\mathrm{ec}B \\ a_{mn}\dot{\mathrm{v}}\mathrm{ec}B \end{pmatrix}$$

$$= \mathrm{vec}(a_{11}\mathrm{vec}B,..,a_{1n}\mathrm{vec}B,..,a_{m1}\mathrm{vec}B,..,a_{mn}\mathrm{vec}B)$$

$$= \mathrm{vec}\{(\mathrm{vec}B)(\mathrm{vec}A')'\},$$

but also

$$\mathrm{vecb}_r C = \mathrm{vec}(a_{11}B,..,a_{1n}B,..,a_{m1}B,..,a_{mn}B) = \mathrm{vec}\{(\mathrm{vec}A')'\otimes B\}.$$

As a second result we have that, if $C = (\mathrm{vec}B)(\mathrm{vec}A)'$, where B is elementwise partitioned, then

$$X = \mathrm{vec}B \otimes \mathrm{vec}A \tag{24}$$

is a solution for the equation $\mathrm{vec}X = \mathrm{vecb}_r C$. This can be seen as follows:

$$C = (\mathrm{vec}B)(\mathrm{vec}A)' = \begin{pmatrix} b_{11}(\mathrm{vec}A)' \\ b_{m1}(\dot{\mathrm{v}}\mathrm{ec}A)' \\ b_{1n}(\dot{\mathrm{v}}\mathrm{ec}A)' \\ b_{mn}(\dot{\mathrm{v}}\mathrm{ec}A)' \end{pmatrix},$$

say. Then

$$\mathrm{vecb}_r C = \begin{pmatrix} b_{11}\mathrm{vec}A \\ b_{m1}\dot{\mathrm{v}}\mathrm{ec}A \\ b_{1m}\dot{\mathrm{v}}\mathrm{ec}A \\ b_{mn}\dot{\mathrm{v}}\mathrm{ec}A \end{pmatrix} = \mathrm{vec}B \otimes \mathrm{vec}A = \mathrm{vec}(\mathrm{vec}B \otimes \mathrm{vec}A).$$

4 APPLICATIONS

In this section we will give some examples where the tilde operator and both block Kronecker products play an important role. Various results concerning the block Kronecker product will be used.

4.1 Approximation of a large matrix

In the context of covariance structure analysis, it can be computationally attractive to approximate a $(k^2 x k^2)$ matrix W by $A \otimes A$ (see Koning, Neudecker and Wansbeek [3]). The criterion to be minimized is

$$\mathrm{tr}\left[A\otimes A - W\right]^2, \tag{25}$$

where A is a symmetric (kxk) matrix as yet to be determined and W is a symmetric $(k^2 x k^2)$ matrix of fourth-order moments, hence $K_{kk}W = WK_{kk} = W$. Denoting the criterion function by $k(a)$, with $a \equiv \mathrm{vec}A$, we have

$$\begin{aligned} k(a) &= \mathrm{tr}\left[(A\otimes A) - W\right]^2 = (a\otimes a)'B'B(a\otimes a) - 2(a\otimes a)'B'W + \mathrm{tr}W^2 \\ &= (a'a)^2 - 2a'\tilde{W}a + \mathrm{tr}W^2 = \left(\mathrm{tr}A^2\right)^2 - 2a'\tilde{W}a + \mathrm{tr}W^2 \end{aligned} \tag{26}$$

with $B = I_k \otimes K_{kk} \otimes I_k$. For this function we derive the first differential. In the derivation, the matrix D is the duplication matrix (see Magnus and Neudecker [5]), and $\mathfrak{v}(A)$ is the vector with nonduplicated elements of $\mathrm{vec}A$. The first differential of $k(a)$ is

$$\mathrm{d}k(A) = 4(\mathrm{tr}A^2)a'\mathrm{d}a - 4a'\tilde{W}\mathrm{d}a = 4(\mathrm{tr}A^2)a'D\mathrm{d}\mathfrak{v}(A) - 4a'\tilde{W}D\mathrm{d}\mathfrak{v}(A).$$

since

$$D\mathfrak{v}(A) = \mathrm{vec}A = a.$$

The first-derivative is now

$$\frac{\partial k(A)}{\partial \mathfrak{v}(A)} = 4(\mathrm{tr}A^2)D'\mathrm{vec}A - 4D'\tilde{W}\mathrm{vec}A,$$

from which we obtain the first order condition

$$(\mathrm{tr}A^2)D'\mathrm{vec}A = D'\tilde{W}\mathrm{vec}A,$$

which leads to

$$(\mathrm{tr}A^2)\mathrm{vec}A = \tilde{W}\mathrm{vec}A. \tag{27}$$

The first-order condition of the minimization problem (25) thus takes the form of an eigenvalue problem in terms of $\tilde{W}$. It is easily seen that the vector a^* minimizing (26) must be proportional to the eigenvector corresponding to the largest eigenvalue. In fact, if λ^* is the largest eigenvalue of $\tilde{W}$ and x^* is the corresponding eigenvector, $a^* = \sqrt{\lambda^*} x^*$ minimizes (26).

4.2 k-factorial covariance structures

In the estimation of k-factorial covariance structures, the distance between the $l \times l$ sample covariance matrix S and the population covariance matrix Ω is minimized. The matrix Ω is restricted to have the form

$$\Omega = \Sigma_1 \otimes \Sigma_2 \otimes \ldots \otimes \Sigma_k,$$

with Σ_i of order $(l_i \times l_i)$. Of course, $\Pi_{i=1}^{k} l_i = l$. A general form of the criterion function used in this problem is

$$q(\theta) = \mathrm{tr}\left[(S - \Omega) W^{-1}\right]^2, \tag{28}$$

with q the function to be minimized, θ the vector of parameters to be estimated (*i.e.*, the elements of Σ_i, $i = 1,..,k$) and W is a weighting matrix.

To derive estimators for Σ_i, some additonal notation is needed. Define

$$W_i \equiv C_i W C_i',$$

with $C_i \equiv I_{l_1 \times \cdots \times l_{i-1}} \otimes K_{l_{i+1} \times .. \times l_k, l_i}$. This matrix C_i is a generalization of the commutation matrix and is discussed extensively in Kapteyn, Neudecker and Wansbeek [2]. The matrices S_i and Ω_i are defined analogously. Note that

$$\Omega_i \equiv C_i \Omega C_i' = \left(\Sigma_1 \otimes .. \otimes \Sigma_{i-1} \otimes \Sigma_{i+1} \otimes .. \otimes \Sigma_k\right) \otimes \Sigma_i \equiv \Sigma^i \otimes \Sigma_i,$$

with Σ^i implicitly defined. Furthermore, define $l^i \equiv \Pi_{j \neq i}^{k} l_j$, $\sigma_i \equiv \mathrm{vec}\Sigma_i$ and $\sigma^i \equiv \mathrm{vec}\Sigma^i$. Finally, we will use the matrix B_i, which is given by $B_i = I_{l^i} \otimes K_{l^i, l_i} \otimes I_{l_i}$. Now we are in a position to evaluate the criterion function:

$$\begin{aligned} q(\theta) &= \mathrm{tr}\left[(S - \Omega) W^{-1}\right]^2 = \mathrm{tr}\left[(S_i - \Omega_i) W_i^{-1}\right]^2 \\ &= \left(\mathrm{vec}(S_i - \Omega_i)\right)' (W_i \otimes W_i)^{-1} \left(\mathrm{vec}(S_i - \Omega_i)\right) \\ &= \left(\mathrm{vec}(S_i - \Omega_i)\right)' B_i' B_i (W_i \otimes W_i)^{-1} B_i' B_i \left(\mathrm{vec}(S_i - \Omega_i)\right) \\ &= \left(\mathrm{vec} S_i - X_i \sigma_i\right)' (W_i \tilde{\pi} W_i)^{-1} \left(\mathrm{vec} S_i - X_i \sigma_i\right), \end{aligned} \tag{28'}$$

with $X_i \equiv \sigma'\otimes I_{l_i}\otimes I_{l_i}$. In this derivation, we have used equation (11) and the property of the tilde operator in equation (7). An estimator for σ_i is thus given by

$$\hat{\sigma}_i = \left[X_i'\left(W_i\pi W_i'\right)^{-1}X_i\right]^{-1}X_i'\left(W_i\pi W_i'\right)^{-1}\mathrm{vec}\tilde{S}_i. \tag{29}$$

A more elaborate discussion of this model can be found in Verhees [10] and Wansbeek and Verhees [12].

4.3 The Moore-Penrose inverse of a design matrix in balanced ANOVA

Consider the usual linear balanced ANOVA model with fixed effects:

$$y_{ij} = \alpha + \beta_{i.} + \gamma_{.j} + \varepsilon_{ij}, \qquad \begin{aligned} i &= 1,..,n \\ j &= 1,..,m \end{aligned} \tag{30}$$

with ε_{ij} a random disturbance term. In matrix format, this can be written as

$$\begin{aligned} y &= \left(\iota_m\otimes\iota_n \;\; I_m\otimes\iota_n \;\; \iota_m\otimes I_n\right)\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} + \varepsilon \\ &\equiv X\theta + \varepsilon. \end{aligned} \tag{31}$$

Estimation of θ requires the Moore-Penrose inverse of the design matrix X. It is convenient, however, to consider a larger matrix $\bar{X}$:

$$\bar{X} \equiv \left(I_m\otimes I_n \;\; \iota_m\otimes I_n \;\; I_m\otimes\iota_n \;\; \iota_m\otimes\iota_n\right).$$

If we define

$$\bar{X}_m \equiv \left(I_m, \iota_m\right),$$

then

$$\bar{X} = \left(I_m, \iota_m\right)\Box\left(I_n, \iota_n\right) = \bar{X}_m\Box\bar{X}_n. \tag{32}$$

The Moore-Penrose inverse is derived by utilizing a singular value decomposition of $\bar{X}$. If $\bar{X}$ has a singular value decomposition $\bar{X} = UDV'$, then the Moore-Penrose inverse is given by $\bar{X}^+ = VD^{-1}U'$. Of course, a singular value decomposition can serve other purposes besides deriving its Moore-Penrose inverse, like assessing the rank of X or to characterize estimable functions of θ. To derive a singular value decomposition of $\mathcal{X}$, we need one preliminary result (see Wansbeek [11]). A singular value decomposition of $\mathcal{X}_m$ is given by

$$\mathcal{X}_m = U_m D_m V_m' \equiv \left(G_m, m^{-1/2}\iota_m\right) \begin{bmatrix} I_{m-1} & 0 \\ 0 & (m+1)^{1/2} \end{bmatrix} \begin{bmatrix} G_m' & 0 \\ \phi_m \iota_m' & m\phi_m \end{bmatrix}, \tag{33}$$

with G_m an mx$(m-1)$ matrix such that $G_m'G_m = I_{m-1}$ and $G_m'\iota_m = 0$. The constant ϕ_m is defined by $\phi_m \equiv (m(m+1))^{-1/2}$. Now we are in a position to derive a singular value decomposition of $\mathcal{X}$:

$$\begin{aligned} \mathcal{X} &= \mathcal{X}_m \square \mathcal{X}_n \\ &= \left(U_m D_m V_m'\right) \square \left(U_n D_n V_n'\right) \\ &= \left(U_m \square U_n\right) \left(D_m \square D_n\right) \left(V_m \square V_n\right)' \\ &\equiv UDV', \end{aligned} \tag{34}$$

which is evidently a singular value decomposition of $\mathcal{X}$. In deriving this result we have used properties 6 and 7 from section 2.2. A singular value decomposition of X can now be obtained as follows. If we define V_R' to be the matrix V' with the first block of columns deleted (corresponding to X being $\mathcal{X}$ with the first block of columns deleted), it is easily established that

$$V_R'V_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha_1 I_{n-1} & 0 & 0 \\ 0 & 0 & \alpha_2 I_{m-1} & 0 \\ 0 & 0 & 0 & \alpha_3 \end{bmatrix}, \tag{35}$$

with $\alpha_1 = m/(m+1)$, $\alpha_2 = n/(n+1)$ and $\alpha_3 = (m+n+mn)/\{(m+1)(n+1)\}$. Now define Q by:

$$Q \equiv \begin{bmatrix} 0 & 0 & 0 \\ I_{n-1} & 0 & 0 \\ 0 & I_{m-1} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then it is easily verified that a singular value decomposition of X is

$$X = (UQ)\left(Q'D[V_R'V_R]^{1/2}Q\right)\left(Q'\left([V_R'V_R]^{+}\right)^{1/2}V_R'\right). \tag{36}$$

See Wansbeek [11] for more details. This approach can be generalized to more complicated models.

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